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Note

On the density of sets of divisors

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Abstract

Consider the lattice of divisors of n, $[1,n]$. For any downset (ideal) $\mathscr{I}$ in $[1,n]$ we get a forbidden configuration theorem of the type that if a set of divisors D avoids certain configurations, then $|D| \leqslant |\mathscr{I}|$. If we let $\mathscr{S}$ be the set of minimal elements of $[1,n]$ not in $\mathscr{I}$, then we forbid in D the configurations $C(s)$ (defined in the paper) for $s \in \mathscr{S}$. This generalizes a result of Alon and in turn generalizes a result of Sauer, Perles and Shelah.

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1. Introduction

Let $[m]=\{1,2,\dots,m\}$. For n having a prime factorization $n=p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$, the divisors are in one to one correspondence with the vectors in $[k_1+1]\times[k_2+1]\times\cdots\times[k_m+1]$. For a divisor $s=\prod_{i\in[m]}p_i^{s_i}$, we will also use the vector notation $s=(s_1,s_2,\dots,s_m)$. Let $[1,n]$ denote the lattice of divisors of n. We will be discussing subsets D of $[1,n]$. These subsets can be interpreted in various ways. We could view D as submultisets of a multiset (where element i occurs at most k_i times), each submultiset corresponding to a divisor. Alternatively, D yields a $(k_1+1)\times(k_2+1)\times\cdots\times(k_m+1)$ (0, 1)-matrix in m dimensions with each 1 corresponding to a divisor. Also if we let an $(m; k_1,k_2,\dots,k_m)$-matrix A be a matrix on m rows whose entries in row i belong to $\{0,1,2,\dots,k_i\}$, then if the columns are distinct, each column corresponds to a divisor. We find the divisor notation somewhat easier to use.

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A downset $\mathscr{I}$ (ideal) in $[1, n]$ is a subset so that if $a \in \mathscr{I}$ and $b|a$ then $b \in \mathscr{I}$. Let $\mathscr{I}$ be any downset in $[1, n]$ and let $\mathscr{S}$ be the set of minimal elements not in $\mathscr{I}$. We follow the shifting idea in [1]. Frankl [4] has popularized the shift operator in extremal set theory. Let a shift operator T act on subsets of $[1, n]$ so that for a subset D of $[1, n]$

(i) $|T(D)| = |D|$,

(ii) $T(D)$ is a downset,

(iii) $T(D)$ does not contain $[1, s]$ for $s \in \mathscr{S}$.

We may conclude that $T(D) \subseteq \mathscr{I}$ and so $|D| \leqslant |\mathscr{I}|$. We must define a specific T and determine the properties of D so that (i)–(iii) hold. We will require that D have no configuration $C(s)$ for $s \in \mathscr{S}$ which we now define informally. Imagine $[1, n]$ as a box of integer grid points in $\mathbb{R}^m$, with axes $x_1, x_2, \ldots, x_m$. Let $s = (s_1, s_2, \ldots, s_m)$. For $m = 3$, $C(s)$ corresponds to $s_3 + 1$ planes parallel to $x_3 = 0$, each plane containing $s_2 + 1$ lines each parallel to $x_2 = x_3 = 0$ and on each line there are $s_1 + 1$ distinct points/divisors. For arbitrary m, $C(s)$ corresponds to $s_m + 1$ $(m-1)$-dimensional subspaces parallel to $x_m = 0$, each $(m-1)$-dimensional subspace containing $s_{m-1} + 1$ $(m-2)$-dimensional subspaces parallel to $x_{m-1} = x_m = 0$, etc., and in each one-dimensinal subspace (parallel to $x_2 = x_3 = \cdots = x_m = 0$) there are $s_1 + 1$ points/divisors.

Let us state Alon's result [1] to see how we are generalizing it. Let $\mathscr{S}_A$ (subscript A for Alon) be a family of subsets of $[m]$. Let A be a matrix on m rows and for $S \subseteq [m]$, let $A_{|S}$ be the submatrix of A consisting of those rows of A indexed by S.

Theorem 1 (Alon [1]). *Let A be an $(m; k_1, k_2, \ldots, k_m)$-matrix with n distinct columns. Assume for no $S \in \mathscr{S}_A$ that $A_{|S}$ contains $\prod_{i \in S}(k_i + 1)$ distinct columns. Then if we let*

$$\mathscr{S} = \left\{ s \in [1, n]: s = \prod_{i \in S} p_i^{k_i} \text{ for some } S \in \mathscr{S}_A \right\},$$

and $\mathscr{I}$ be the downset in $[1, n]$ for which $\mathscr{S}$ is the set of minimal elements not in $\mathscr{I}$, then

$$n \leqslant |\mathscr{I}|.$$

Note that $\mathscr{S}$ and hence $\mathscr{I}$ have a special structure, nonetheless, our forbidden configurations $C(s)$ for $s \in \mathscr{S}$ correspond to the same restriction on A. Our result (Theorem 5) extends Theorem 1 (which in turn generalizes a basic results of Sauer [5] and Perles and Shelah [6]) to an arbitrary ideal. Some other examples where $\mathscr{I}$ has special structure and better forbidden configurations can be determined are explored in [3].

2. Main results

We define T via the shift operator of Alon. Let

$$\bar{T}_{i,j}(a_1, a_2, \ldots, a_i, \ldots, a_m) = \begin{cases} (a_1, a_2, \ldots, a_i - 1, \ldots, a_m) & \text{if } a_i = j, \\ (a_1, a_2, \ldots, a_i, \ldots, a_m) & \text{otherwise.} \end{cases}$$

We then let

$$T_{i,j}(\boldsymbol{a}) = \begin{cases} \bar{T}_{i,j}(\boldsymbol{a}) & \text{if } \bar{T}_{i,j}(\boldsymbol{a}) \notin D, \\ \boldsymbol{a} & \text{otherwise,} \end{cases}$$

and extend naturally to define $T_{i,j}(D)$.

We define

$$T_i = (T_{i,1} \circ T_{i,2} \circ \cdots \circ T_{i,k_i}) \circ \cdots \circ (T_{i,1} \circ T_{i,2}) \circ T_{i,1},$$

i.e. shift as much as possible in the ith coordinate. Finally the shift operator T is defined as follows:

$$T = T_m \circ T_{m-1} \circ \cdots \circ T_2 \circ T_1.$$

Lemma 2. *Let D be a subset of $[1, n]$. Then $T(D)$ is a downset.*

The proof is postponed until later in the paper.

It is useful to talk about certain projections of sets of divisors. For $J \subseteq [m]$, let

$$D_{|J} = \left\{ \prod_{j \in J} p_j^{d_j} \colon \boldsymbol{d} = (d_1, d_2, \ldots, d_m) \in D \right\}.$$

Now for any $\boldsymbol{e} = (e_1, \ldots, e_m) \in D$ let

$$D_{|J,\boldsymbol{e}} = \left\{ \prod_{j \in J} p_j^{d_j} \colon \boldsymbol{d} = (d_1, d_2, \ldots, d_m) \in D,\ d_i = e_i \text{ for } i \notin J \right\}.$$

Lemma 3 (Alon [1]). *For each $J \subseteq [m]$, $|T_{i,j}(D)_{|J}| \leqslant |D_{|J}|$ with equality for $J = [m]$.*

Under our shift operator we can determine which configurations in D give rise to $[1, s]$ in $T(D)$, for $s = \prod_{i \in [m]} p_i^{s_i} \in \mathscr{S}$. Define the configuration $C(s)$ of divisors in D inductively as follows. Let $s' = \prod_{i \in [m-1]} p_i^{s_i}$. The configuration $C(s)$ consists of $s_m + 1$ different values of $0 \leqslant i_1 < i_2 < \cdots i_{s_m+1} \leqslant k_m$ so that, defining $\boldsymbol{i}_t = (0, \ldots, 0, i_t)$, in each $D|_{[m-1], \boldsymbol{i}_t}$ we have the configuration $C(s')$.

Lemma 4. *If D has no configuration $C(s)$ for $s \in \mathscr{S}$, then neither does $T(D)$.*

We delay the proof until later in the paper.

We can now state our main result.

Theorem 5. *Let $\mathscr{I}$ be a downset in the lattice of divisors $[1, n]$ and let $\mathscr{S}$ be the set of minimal elements of $[1, n]$ not in $\mathscr{I}$. Let D be a set of divisors of n with no configuration $C(s)$ for $s \in \mathscr{S}$. Then*

$$|D| \leqslant |\mathscr{I}|.$$

Proof. By Lemma 2, $T(D)$ is a downset. By Lemma 3 (or directly), $|T(D)|=|D|$. By Lemma 4, $T(D)$ does not contain $[1,s]$ for $s\in\mathscr{S}$ since D does not have $C(s)$. So $T(D)\subseteq\mathscr{I}$ and the result follows. $\square$

Because we have obtained a result for any downset $\mathscr{I}$, configuration theorems follow in profusion. We can for example obtain results which yield a bound on D that is polynomial in m such as the following result of Woodall.

Theorem 6 (Woodall, see Problem 10.10 [2]). *Let D be a set of divisors of n such that for x, $y\in D$ and z any product of l prime factors (not necessarily distinct),*

$$z \text{ does not divide } \gcd(x,y); \tag{1}$$

then

$$|D|\leqslant\binom{k_1+k_2+\cdots+k_m+m}{l}+\binom{k_1+k_2+\cdots+k_m+m}{l-1}+\cdots+\binom{k_1+k_2+\cdots+k_m+m}{0}. \tag{2}$$

Proof. The configuration $C(s)$ implies the existence of numbers with large common factors. Let $\mathrm{Trim}(s)=s/p_i$, where $i=\min\{j\colon p_j \text{ divides } s\}$. Now, if D contains $C(s)$ then D contains two divisors x, y with $\mathrm{Trim}(s)$ a divisor of $\gcd(x,y)$. Now apply Theorem 5 with $\mathscr{I}$ being all possible products in $[1,n]$ with exactly l factors and so $\mathscr{S}$ is all possible products in $[1,n]$ with at most $l+1$ factors. Thus our condition (1) on D implies that D has no $C(s)$ for $s\in\mathscr{S}$. $\square$

We can interpret Theorem 5 in terms of an $(m;k_1+1,k_2+1,\ldots,k_m+1)$-matrix A. If A has n distinct columns and no configuration $C(s)$ for $s\in\mathscr{S}$, then $n\leqslant|\mathscr{I}|$. But what is $C(s)$? Let $s=p_1p_2p_3$ for example. Then $C(s)$ corresponds to a row and column permutation of any 3×8 submatrix in rows 1, 2, 3 of A of the form

$$\begin{pmatrix} a & a' & b & b' & c & c' & d & d' \\ e & e & e' & e' & f & f & f' & f' \\ g & g & g & g & g' & g' & g' & g' \end{pmatrix}, \tag{3}$$

where the restrictions on the entries are $a\neq a'$, $b\neq b'$, $\ldots$, $g\neq g'$. At this point we do not have any applications of our general result in this setting but we note that, for example, if we require any square submatrix of A to have determinant in $\{-1,0,1\}$, then A has no configuration $C(p_1p_2p_3)$, since the entries of (3) are forced to be 0, 1 and so we get a 3×3 submatrix of determinant ± 2.

3. Proofs of lemmas

Let us now return to fill in the proofs deferred earlier.

Proof of Lemma 2. We use induction on m to show that if $\bar{D}=T_i\circ T_{i-1}\circ\cdots\circ T_1(D)$, then $\bar{D}|_{[i],e}$ is a downset for each possible e. We define $\hat{D}=T_{i-1}\circ\cdots\circ T_1(D)$, and assume that $\hat{D}|_{[i-1],e}$ is a downset for each e and show that $T_{i,j}(\hat{D})|_{[i-1],e}$ is a downset for each e.

Let $a=(a_1,\dots,a_i,\dots,a_m)\in T(D)$ and let a^* be such that $T_{i,j}(a^*)=a$, $a^*=(a_1,\dots,a_i^*,\dots,a_m)$.

Case 0: $a_i^*\neq j$. The result follows by induction.

Case 1: $a_i^*=j$ and $a_i=j$. Now $T_{i,j}(a)=a$ so $(a_1,a_2,\dots,j-1,\dots,a_m)\in D$. $\hat{D}|_{[i-1],e}$, $e=(0,\dots,0,j-1,\dots,a_m)$, contains $[1,\prod_{l\in[i-1]}p_l^{a_l}]$ and as such is fixed by $T_{i,j}$. Hence $T_{i,j}(\hat{D})|_{[i-1],a}$ contains $[1,\prod_{l\in[i-1]}p_l^{a_l}]$.

Case 2: $a_i^*=j$, $a_i=j-1$. By the inductive hypothesis, $\hat{D}|_{i-1],a^*}$ contains $[1,\prod_{l\in[i-1]}p_l^{a_l}]$. For each $a'=(a_1',a_2',\dots,a_{i-1}',j,a_{i+1},\dots,a_m)\in\hat{D}$ and so $a''\in T_{i,j}(\hat{D})$ or $T_{i,j}(a')=a''$. Thus $T_{i,j}(\hat{D})|_{[i-1],a}$ contains $[1,\prod_{l\in[i-1]}p_l^{a_l}]$. Now, $T_i(\hat{D})=\bar{D}$, and T_i is a composition of $T_{i,j}$ operators. Hence, inductively, $\bar{D}|_{[1,i-1],e}$ is a downset for each e. But $(a_1,\dots,a_i,\dots,a_m)\in\bar{D}$ implies $(a_1,\dots,a_i',\dots,a_m)\in\bar{D}$ for $0\leqslant a_i'\leqslant a_i$ (by the properties of T_i, and since $T_i(\hat{D})=\bar{D}$) which implies $(a_1',\dots,a_i',\dots,a_m)\in\bar{D}$ for all $1\leqslant a_j'\leqslant a_j$, for all $1\leqslant j\leqslant i$, by properties of downsets. So $T_i\circ T_{i-1}\circ\cdots\circ T_1(D)|_{[i],e}$ is a downset for every e. Now the result follows by induction when $i=m$. $\square$

Proof of Lemma 4. The proof uses induction on m. Assume that $T(D)$ has $[1,s]$ for $s=\prod_{i\in[m]}p_i^{s_i}$ so $p_m^k\prod_{i\in[m-1]}p_i^{s_i}\in T(D)$ for $0\leqslant k\leqslant s_m$. Now if we let $D'=T_i\circ T_{i-1}\circ T_1(D)$, then $T(D)=T_m(D)$. Thus there exist $i_0<i_1<\cdots<i_{s_m}$ so that $p_m^{i_k}\prod_{i\in[m-1]}p_i^{s_i}\in D'$ for $0\leqslant k\leqslant s_m$. But now, using Lemma 1, $D|_{[m-1],e}$ is a downset and so, if we let $s'=\prod_{i\in[m-1]}p_i^{s_i}$, then for each $x\in[1,s']$, we have $xp_m^{i_k}\in D'$ for $0\leqslant k\leqslant s_m$. Now apply induction to get that for each of the s_m+1 values i_k, $D|_{[m-1],i_k}$ contains $C(s')$ and so D contains $C(s')$. $\square$

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